MTH 408/522: Numerical analysis

Homework I: Bisection and Fixed-point iteration methods

(Due 02/09/19)

Problems for turning in

Bisection method

1. Show that the order of convergence of the Bisection method is sublinear.
2. Find a bound on the number of iterations needed to achieve an approximation with 10^{-3} to the solution of $x^3 + x - 4 = 0$ lying in the interval $[1, 4]$.
3. Let $f(x) = (x-1)^{10}$, $p = 1$, and $p_n = 1 + 1/n$. Show that $|f(p_n)| < 10^{-3}$, whenever $n > 1$, but that $|p - p_n| < 10^{-3}$ requires that $n > 1000$.

Fixed-point iteration method

4. Consider $g(x) = 2x - Ax^2$, where $A > 0$.
   (a) Show that if the sequence $\{p_n\}$ generated by g converges, then $p_n \to 1/A$.
   (b) Find an interval about $1/A$ for which p_n converges.
5. Suppose that g is a continuously differentiable function on some interval (c, d) that contain the fixed point of g. Show that if $|g'(p)| < 1$, then there exists $\delta > 0$ such that if $|p_0 - p| \leq \delta$, then $\{p_n\}$ generated by g converges.
6. Show that if $A > 0$, then the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}}, \text{ for } n \geq 1,$$

converges to $\sqrt{A}$, whenever $x_0 > 0$. What happens if $x_0 < 0$?

Problems for practice

1. In each of the following, use the Bisection method to find solutions to the equation $f(x) = 0$ in the interval $[a, b]$ accurate to within ACC.
   (a) $f(x) = x^2 - 4x + 4 - \ln(x)$; $[a, b] = [2, 4]$; $ACC = 10^{-5}$.
   (b) $f(x) = x + 1 - 2\sin(\pi x)$; $[a, b] = [0, 0.5]$; $ACC = 10^{-5}$.
   (c) $f(x) = x^3 - 7x^2 + 14x - 6$; $[a, b] = [3.2, 4]$; $ACC = 10^{-2}$.
   (d) $f(x) = e^x - x^2 + 3x - 2$; $[a, b] = [0, 1]$; $ACC = 10^{-2}$.
   (e) $f(x) = e^x - \cos(e^x - 2) - 2$; $[a, b] = [0.5, 1.5]$; $ACC = 10^{-2}$.
2. In each of the following, use the Fixed-point Iteration method to find solutions to the equation $f(x) = x$ accurate to within ACC in the interval $[a, b]$ (or after determining a suitable interval $[a, b]$ in which a root exists).
   (a) $f(x) = 6^{-x}$; $ACC = 10^{-5}$.
   (b) $f(x) = (e^x/3)^{1/2}$; $ACC = 10^{-5}$.

(c) $f(x) = x^3 - 2x - 5$; $[a, b] = [2, 3]$; $ACC = 10^{-2}$.

(d) $f(x) = x^2 + 10\cos(x)$; $ACC = 10^{-4}$.

(e) $f(x) = 3x^2 - e^x$; $ACC = 10^{-2}$.